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Special Types of Locally Conformal Closed G_2-Structures

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Abstract: Motivated by known results in locally conformal symplectic geometry, we study different classes of G_2-structures defined by a locally conformal closed 3-form. In particular, we provide a complete characterization of invariant exact locally conformal closed G_2-structures on simply connected Lie groups, and we present examples of compact manifolds with different types of locally conformal closed G_2-structures.

Keywords: locally conformal closed G_2-structure; coupled SU(3)-structure

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1. Introduction

Over the last years, the study of smooth manifolds endowed with geometric structures defined by a differential form which is locally conformal to a closed one has attracted a great deal of attention. Particular consideration has been devoted to *locally conformal Kähler* (*LCK*) structures and their non-metric analogous, *locally conformal symplectic* (*LCS*) structures (see [1–5] and the references therein). In both cases, the condition of being locally conformal closed concerns a suitable non-degenerate 2-form ω and is encoded in the equation $d\omega = \theta \wedge \omega$, where θ is a closed 1-form called the *Lee form*. LCK structures belong to the pure class $\mathcal{W}_4$ of Gray–Hervella's celebrated 16 classes of almost Hermitian manifolds (see [6]). They are, in particular, Hermitian structures, and their understanding on compact complex surfaces is related to the global spherical shell conjecture of Nakamura. As pointed out in [5], LCS geometry is intimately related to Hamiltonian mechanics. Very recently, Eliashberg and Murphy used h-principle arguments to prove that every almost complex manifold M with a non-zero $[\theta] \in H^1_{dR}(M)$ admits an LCS structure whose Lee form is (a multiple of) θ (see [7]).

In odd dimensions, 7-manifolds admitting G_2-structures provide a natural setting where the locally conformal closed condition is meaningful. Recall that G_2 is one of the exceptional Riemannian holonomy groups resulting from Berger's classification [8], and that a G_2-structure on a 7-manifold M is defined by a 3-form φ with a pointwise stabilizer isomorphic to G_2. Such a 3-form gives rise to a Riemannian metric g_φ and to a volume form dV_φ on M, with corresponding Hodge operator $*_\varphi$. By an h-principle argument, it is possible to show that every compact 7-manifold admitting G_2-structures always admits a coclosed G_2-structure, i.e., one whose defining 3-form φ fulfills $d *_\varphi \varphi = 0$ [9]. A G_2-structure φ satisfying the conditions

$$d\varphi = \theta \wedge \varphi, \quad d *_\varphi \varphi = \frac{4}{3}\,\theta \wedge *_\varphi \varphi \tag{1}$$

for some closed 1-form θ is locally conformal to one which is both closed and coclosed. G_2-structures fulfilling Equation (1) correspond to the class $\mathcal{W}_4$ in Fernández–Gray's classification [10], and they are called *locally conformal parallel* (*LCP*), as being closed and coclosed for a G_2-form φ is equivalent to being parallel with respect to the associated Levi Civita connection (see [10]). It was proved by Ivanov, Parton, and Piccinni in [11] (Theorem A) that a compact LCP G_2-manifold is a mapping torus bundle over $\mathbb{S}^1$ with fiber a simply connected nearly Kähler manifold of dimension six and finite structure group. This shows that LCP G_2-structures are far from abundant.

Relaxing the LCP requirement by ruling out the second condition in Equation (1) leads naturally to *locally conformal closed*, a.k.a. *locally conformal calibrated* (*LCC*), G_2-structures. Additionally, the unique closed 1-form θ for which $d\varphi = \theta \wedge \varphi$ is called the *Lee form*. LCC G_2-structures have been investigated in [12–14]; in particular, in [12], the authors showed that a result similar to that of Ivanov, Parton, and Piccinni holds for compact manifolds with a suitable LCC G_2-structure. Roughly speaking, they are mapping tori bundle over $\mathbb{S}^1$ with fiber a 6-manifold endowed with a *coupled* $\mathrm{SU}(3)$-structure, of which nearly Kähler structures constitute a special case. We refer the reader to Theorem 1 below for the relevant definitions and the precise statement.

In LCS geometry, one distinguishes between structures of the first kind and of the second kind (see [5,15]); the distinction depends on whether or not one can find an infinitesimal automorphism of the structure, which is transversal to the foliation defined by the kernel of the Lee form. The geometry of an LCS structure of the first kind is very rich and is related to the existence of a contact structure on the leaves of the corresponding foliation (cf. [1,15]). Another way to distinguish LCS structures is according to the vanishing of the class of ω in the Lichnerowicz cohomology defined by the Lee form. This leads to the notions of exact and non-exact LCS structures. An LCS structure of the first kind is always exact, but the converse is not true (see, e.g., [15] (Example 5.4)). The LCS structures constructed by Eliashberg and Murphy in [7] are exact.

The purpose of this note is to bring ideas of LCS geometry into the study of LCC G_2-structures. In Sections 3 and 4, after recalling the notion of conformal class of an LCC G_2-structure, we consider exact structures, and we distinguish between structures of the first and of the second kind. As it happens in the LCS case, the difference between first and second kind depends on the existence of a certain infinitesimal automorphism of the LCC G_2-structure φ, which is everywhere transversal to the kernel of the Lee form. As for exactness, every LCC G_2-structure φ defines a class $[\varphi]_\theta$ in the Lichnerowicz cohomology $H^\bullet_\theta(M)$ associated with the Lee form θ; φ is said to be exact if $[\varphi]_\theta = 0 \in H^3_\theta(M)$. As we shall see, LCC G_2-structures of the first kind are always exact, but the opposite does not need to be true (cf. Example 3). It is an open question whether an h-principle argument can be used to prove the existence of an exact LCC G_2-structure on a compact manifold admitting G_2-structures.

In the literature, there exist many examples of left-invariant LCP and LCC G_2-structures on solvable Lie groups (see e.g., [12,14,16]). In the LCC case, the examples exhibited in [12] admit a lattice and hence provide compact solvmanifolds endowed with an invariant LCC G_2-structure. In Section 5, we completely characterize the left-invariant exact LCC G_2-structures on simply connected Lie groups: their Lie algebra is a rank-one extension of a six-dimensional Lie algebra with a coupled $\mathrm{SU}(3)$-structure by a suitable derivation (see Theorem 2). Moreover, using the classification of seven-dimensional nilpotent Lie algebras carrying a closed G_2-structure [17], we prove that no such nilpotent Lie algebra admits an LCC G_2-structure (Proposition 5). Finally, in Section 6, we show that there exist solvable Lie groups admitting a left-invariant LCC G_2-structure, which is not exact (see Example 1). This does not happen on nilpotent Lie groups, as every left-invariant LCC G_2-structure must be exact by a result of Dixmier [18] on the Lichnerowicz cohomology. We also show that, unlike the LCS case, there exist exact LCC structures on unimodular Lie algebras that are not of the first kind (see Remark 6).

2. Preliminaries

Let M be a seven-dimensional manifold. A G_2-reduction of its frame bundle, i.e., a G_2-*structure*, is characterized by the existence of a 3-form $\varphi \in \Omega^3(M)$, which can be pointwise written as

$$\varphi|_p = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

with respect to a basis $(e^1, \ldots, e^7)$ of the cotangent space T_p^*M. Here, the notation e^{ijk} is a shorthand for $e^i \wedge e^j \wedge e^k$. A G_2-*structure* φ gives rise to a Riemannian metric g_φ with volume form dV_φ via the identity

$$g_\varphi(X,Y)\, dV_\varphi = \frac{1}{6}\, \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. We shall denote by $*_\varphi$ the corresponding Hodge operator.

When a G_2-structure φ on M is given, the G_2-action on k-forms (cf. [19] (Section 2)) induces the following decompositions:

$$\begin{aligned}
\Omega^2(M) &= \Omega^2_7(M) \oplus \Omega^2_{14}(M) \\
\Omega^3(M) &= \mathcal{C}^\infty(M)\,\varphi \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M)
\end{aligned}$$

where

$$\Omega^2_7(M) := \{\iota_X\varphi | X \in \mathfrak{X}(M)\}, \quad \Omega^2_{14}(M) := \left\{\kappa \in \Omega^2(M) | \kappa \wedge *_\varphi\varphi = 0\right\}$$

$$\Omega^3_7(M) := \left\{*_\varphi(\varphi \wedge \alpha) | \alpha \in \Omega^1(M)\right\}, \ \Omega^3_{27}(M) := \left\{\gamma \in \Omega^3(M) | \gamma \wedge \varphi = 0,\ \gamma \wedge *_\varphi\varphi = 0\right\}.$$

The decompositions of $\Omega^k(M)$, for $k = 4, 5$, are obtained from the previous ones via the Hodge operator.

By the above splittings, on a 7-manifold M endowed with a G_2-structure φ there exist unique differential forms $\tau_0 \in \mathcal{C}^\infty(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$, and $\tau_3 \in \Omega^3_{27}(M)$, such that

$$d\varphi = \tau_0 *_\varphi \varphi + 3\,\tau_1 \wedge \varphi + *_\varphi\tau_3, \quad d *_\varphi \varphi = 4\,\tau_1 \wedge *_\varphi\varphi + \tau_2 \wedge \varphi \tag{2}$$

see [20] (Proposition 1). Such forms are called *intrinsic torsion forms* of the G_2-structure φ, as they completely determine its intrinsic torsion. In particular, φ is *torsion-free* if and only if all of these forms vanish identically, that is, if and only if φ is both *closed* ($d\varphi = 0$) and *coclosed* ($d *_\varphi \varphi = 0$). When this happens, g_φ is Ricci-flat and its holonomy group is isomorphic to a subgroup of G_2.

In this paper, we shall mainly deal with the G_2-structures defined by a 3-form which is locally conformal equivalent to a closed one. As we will see in Section 3, this condition corresponds to the vanishing of the intrinsic torsion forms τ_0 and τ_3. For the general classification of G_2-structures, we refer the reader to [10].

Since G_2 acts transitively on the 6-sphere with stabilizer SU(3), a G_2-structure φ on a 7-manifold M induces an SU(3)-structure on every oriented hypersurface. Recall that an SU(3)-*structure* on a 6-manifold N is the data of an almost Hermitian structure (g, J) with fundamental 2-form $\omega := g(J\cdot, \cdot)$, and a unit $(3,0)$-form $\Psi = \psi + i\hat{\psi}$, where $\psi, \hat{\psi} \in \Omega^3(N)$. By [21], the whole SU(3)-structure (g, J, Ψ) is completely determined by the 2-form ω and the 3-form $\psi = \Re(\Psi)$. In particular, at each point p of N, there exists a basis $(e^1, \ldots, e^6)$ of the cotangent space T_p^*N such that

$$\omega|_p = e^{12} + e^{34} + e^{56}, \quad \psi|_p = e^{135} - e^{146} - e^{236} - e^{245}.$$

In a similar way, as in the case of G_2-structures, the intrinsic torsion of an SU(3)-structure (ω, ψ) is encoded in the exterior derivatives $d\omega$, $d\psi$, $d\hat{\psi}$ (see [22]). According to [22] (Definition 4.1), an SU(3)-structure is called *half-flat* if $d\omega \wedge \omega = 0$ and $d\psi = 0$. A half-flat SU(3)-structure is said to be *coupled* if $d\omega = c\,\psi$, for some $c \in \mathbb{R} \setminus \{0\}$, while it is called *symplectic half-flat* if $c = 0$, that is, if the fundamental 2-form ω is symplectic. We shall refer to c as the *coupling constant*.

If $h : N \hookrightarrow M$ is an oriented hypersurface of a 7-manifold M endowed with a G_2-structure φ, and V is a unit normal vector field on N, then the SU(3)-structure on N induced by φ is defined by the differential forms

$$\omega := h^*(\iota_V \varphi), \quad \psi := h^*\varphi.$$

The reader may refer to [23] for more details on the relationship between G_2- and SU(3)-structures in this setting.

3. Locally Conformal Closed G_2-Structures

A G_2-structure φ on a 7-manifold M is said to be *locally conformal closed* or *locally conformal calibrated* (*LCC* for short) if

$$d\varphi = \theta \wedge \varphi \tag{3}$$

for some $\theta \in \Omega^1(M)$. Notice that such a 1-form is unique and closed, as the map

$$\cdot \wedge \varphi : \Omega^k(M) \to \Omega^{k+3}(M), \quad \alpha \mapsto \alpha \wedge \varphi$$

is injective for $k = 1, 2$. Moreover, it can be written in terms of φ as follows:

$$\theta = -\frac{1}{4} *_\varphi (*_\varphi d\varphi \wedge \varphi)$$

(see [13] (Lemma 2.1)).

Definition 1. *The unique closed 1-form θ fulfilling Equation (3) is called the Lee form of the LCC G_2-structure φ.*

Henceforth, we denote an LCC G_2-structure φ with Lee form θ by (φ, θ). As the name suggests, an LCC G_2-structure (φ, θ) is locally conformal equivalent to a closed one. Indeed, since $d\theta = 0$, each point of M admits an open neighborhood $\mathcal{U} \subseteq M$ where $\theta = df$, for some $f \in \mathcal{C}^\infty(\mathcal{U})$, and the 3-form $e^{-f}\varphi$ defines a closed G_2-structure on $\mathcal{U}$ with associated metric $e^{-\frac{2}{3}f} g_\varphi$ and orientation $e^{-\frac{7}{3}f} dV_\varphi$. Moreover, an LCC G_2-structure is globally conformal equivalent to a closed one when θ is exact, and it is closed if and only if θ vanishes identically.

Given an LCC G_2-structure (φ, θ), we may consider its *conformal class*

$$\left\{ e^{-f}\varphi \mid f \in \mathcal{C}^\infty(M) \right\}.$$

It is easily seen that $\left(e^{-f}\varphi, \theta - df\right)$ is also LCC, so the de Rham class $[\theta] \in H^1_{dR}(M)$ is an invariant of the conformal class.

Remark 1.

(1) *The only non-identically vanishing intrinsic torsion forms of an LCC G_2-structure (φ, θ) are $\tau_1 = \frac{1}{3}\theta$ and $\tau_2 \in \Omega^2_{14}(M)$ (cf. (2)). In particular,*

$$d *_\varphi \varphi = \frac{4}{3}\,\theta \wedge *_\varphi \varphi + \tau_2 \wedge \varphi.$$

When τ_2 vanishes identically, the G_2-structure is called locally conformal parallel *(see [11,16,24] for related results).*

(2) *LCC G_2-structures belong to the class $\mathcal{W}_2 \oplus \mathcal{W}_4$ in Fernández–Gray classification [10]. The subclasses $\mathcal{W}_2$ and $\mathcal{W}_4$ correspond to closed and locally conformal parallel G_2-structures, respectively.*

Simple examples of manifolds admitting an LCC G_2-structure can be obtained as follows. Start with a 6-manifold N endowed with a coupled SU(3)-structure (ω, ψ) such that $d\omega = c\psi$ (various examples can be found, for instance, in [14,25,26]). The product manifold $N \times \mathbb{R}$ then admits an LCC G_2-structure given by the 3-form $\varphi = \omega \wedge dt + \psi$, where dt denotes the global 1-form on $\mathbb{R}$. The Lee form of φ is $\theta = -c\,dt$.

More generally, if (ω, ψ) is coupled and $\nu \in \mathrm{Diff}(N)$ is a diffeomorphism such that $\nu^*\omega = \omega$, then the quotient of $N \times \mathbb{R}$ by the infinite cyclic group of diffeomorphisms generated by $(p, t) \mapsto (\nu(p), t + 1)$ is a smooth seven-dimensional manifold N_ν endowed with an LCC G_2-structure φ (see [12] (Proposition 3.1)). N_ν is called the *mapping torus* of ν, and the natural projection $N_\nu \to \mathbb{S}^1$, $[(p, t)] \mapsto [t]$, is a smooth fiber bundle with fiber N. Notice that $N_{\mathrm{Id}} = N \times \mathbb{S}^1$.

In [13], Fernández and Ugarte proved that the LCC condition can be characterized in terms of a suitable differential subcomplex of the de Rham complex. In detail,

Proposition 1 ([13])**.** *A G_2-structure φ on a 7-manifold M is LCC if and only if the exterior derivative of every 3-form in $\mathcal{B}^3(M) := \mathcal{C}^\infty(M)\varphi \oplus \Omega^3_{27}(M)$ belongs to $\mathcal{B}^4(M) := \Omega^4_7(M) \oplus \Omega^4_{27}(M)$. Consequently, φ is LCC if and only if there exists the complex*

$$0 \to \mathcal{B}^3(M) \xrightarrow{\hat{d}} \mathcal{B}^4(M) \xrightarrow{\hat{d}} \Omega^5(M) \xrightarrow{d} \Omega^6(M) \xrightarrow{d} \Omega^7(M) \to 0$$

where $\hat{d}$ denotes the restriction of the differential d to $\mathcal{B}^k(M)$, for $k = 3, 4$.

As the Lee form θ of an LCC G_2-structure φ is closed, it is also possible to introduce the *Lichnerowicz* (or *Morse–Novikov*) *cohomology* of M relative to θ. This is defined as the cohomology $H^\bullet_\theta(M)$ corresponding to the complex $(\Omega^\bullet(M), d_\theta)$, where

$$d_\theta : \Omega^k(M) \to \Omega^{k+1}(M), \quad d_\theta\alpha := d\alpha - \theta \wedge \alpha.$$

It is clear that Equation (3) is equivalent to $d_\theta\varphi = 0$. Thus, φ defines a cohomology class $[\varphi]_\theta \in H^3_\theta(M)$. If $[\varphi]_\theta = 0$, namely if $\varphi = d_\theta\sigma$ for some $\sigma \in \Omega^2(M)$, then the LCC G_2-structure φ is said to be *d_θ-exact* or *exact*. Notice that being exact is a property of the conformal class of φ.

More generally, if a G_2-structure φ is d_θ-exact with respect to some closed 1-form θ, then it is LCC with Lee form θ. The converse might not be true, as we shall see in Example 1.

4. LCC G_2-Structures of the First and of the Second Kind

A special class of exact LCC G_2-structures can be introduced after some considerations of the infinitesimal automorphisms.

Recall that the *automorphism group* of a seven-dimensional manifold M endowed with a G_2-structure φ is

$$\mathrm{Aut}(M, \varphi) := \{F \in \mathrm{Diff}(M) \mid F^*\varphi = \varphi\}\,.$$

Clearly, $\mathrm{Aut}(M, \varphi)$ is a closed Lie subgroup of the isometry group $\mathrm{Iso}(M, g_\varphi)$ of the Riemannian manifold (M, g_φ). Moreover, its Lie algebra is given by

$$\mathfrak{aut}(M, \varphi) := \{X \in \mathfrak{X}(M) \text{ complete} \mid \mathcal{L}_X\varphi = 0\}\,,$$

and every *infinitesimal automorphism* $X \in \mathfrak{aut}(M, \varphi)$ is a Killing vector field for g_φ.

If φ is closed and $X \in \mathfrak{aut}(M, \varphi)$, then the 2-form $\iota_X\varphi \in \Omega^2_7(M)$ is easily seen to be harmonic. When M is compact, this implies that $\mathfrak{aut}(M, \varphi)$ is Abelian with dimension bounded by $\min\{6, b_2(M)\}$ (see [27]).

Let us now focus on the case when φ is LCC with Lee form θ not identically vanishing. For every infinitesimal automorphism $X \in \mathfrak{aut}(M, \varphi)$, we have

$$0 = d(\mathcal{L}_X\varphi) = \mathcal{L}_X d\varphi = \mathcal{L}_X\theta \wedge \varphi$$

hence we see that $\mathcal{L}_X\theta = 0$. Consequently, $\theta(X)$ is constant, and the map

$$\ell_\theta : \mathfrak{aut}(M, \varphi) \to \mathbb{R}, \quad \ell_\theta(X) := \theta(X)$$

is a well-defined morphism of Lie algebras. This suggests that various meaningful ideas of locally conformal symplectic geometry (e.g., [1,5,15,28]) make sense for LCC G_2-structures, too. In particular, as the map ℓ_θ is either identically zero or surjective, we give the following G_2-analogue of a definition first introduced by Vaisman in [5].

Definition 2. *An LCC G_2-structure (φ, θ) is of the first kind if the Lie algebra morphism ℓ_θ is surjective, while it is of the second kind otherwise.*

If there exists at least one point p of M where $\theta|_p = 0$, then the LCC G_2-structure φ is necessarily of the second kind. As a consequence, if φ is an LCC G_2-structure with Lee form θ such that $\theta|_p = df|_p$ for some smooth function $f \in \mathcal{C}^\infty(M)$, then the 3-form $e^{-f}\varphi$ defines an LCC G_2-structure of the second kind, as the corresponding Lee form is $\theta - df$. Hence, being of the first kind is not an invariant of the conformal class of φ.

Assume now that φ is an LCC G_2-structure of the first kind. Then, its Lee form θ is nowhere vanishing; consequently, $\chi(M) = 0$ if M is compact. Let us consider an infinitesimal automorphism $U \in \mathfrak{aut}(M, \varphi)$ such that $\theta(U) = -1$. The condition $\mathcal{L}_U\varphi = 0$ is equivalent to

$$\varphi = d\sigma - \theta \wedge \sigma$$

where $\sigma := \iota_U\varphi \in \Omega^2_7(M)$. Thus, an LCC G_2-structure of the first kind is always exact. More precisely, it belongs to the image of the restriction of d_θ to $\Omega^2_7(M)$.

Remark 2.

(1) *Comparing our situation to the LCS case [5], we are choosing the opposite sign for the infinitesimal automorphism U. This is only a matter of convention and simplifies our presentation.*

(2) *As we mentioned above, if (ω, ψ) is a coupled $\mathrm{SU}(3)$ structure on a 6-manifold N and $\nu \in \mathrm{Diff}(N)$ satisfies $\nu^*\omega = \omega$, then the mapping torus N_ν of ν admits an LCC G_2-structure (φ, θ). It follows from the proof of [12] (Proposition 3.1) that there exists an infinitesimal automorphism $X \in \mathfrak{aut}(N_\nu, \varphi)$ such that $\theta(X) \neq 0$. Thus, (φ, θ) is of the first kind.*

We shall say that an exact G_2-structure φ is *of the first kind* if it can be written as $\varphi = d_\theta(\iota_X\varphi)$ with $\theta(X) = -1$.

Proposition 2. *Let (φ, θ) be an LCC G_2-structure. Then, $\varphi = d_\theta(\iota_X\varphi)$ if and only if $\mathcal{L}_X\varphi = (\theta(X) + 1)\varphi$. In particular, φ is of the first kind if and only if $\theta(X) = -1$.*

Proof. The first assertion follows from the identity

$$d_\theta(\iota_X\varphi) = d(\iota_X\varphi) - \theta \wedge \iota_X\varphi = \mathcal{L}_X\varphi - \theta(X)\varphi.$$

The second assertion is an immediate consequence of the above definition. □

Some examples of LCC G_2-structures of the first and of the second kind will be discussed in Section 6. In particular, we will see that there exist exact G_2-structures of the form $\varphi = d_\theta\sigma$ with $\sigma \notin \Omega^2_7(M)$.

In [12] (Theorem 6.4), the structure of compact 7-manifolds admitting an LCC G_2-structure satisfying suitable properties was described. In view of the definitions introduced in this section, we can rewrite the statement of this structure theorem as follows.

Theorem 1 ([12])**.** *Let M be a compact seven-dimensional manifold endowed with an LCC G_2-structure (φ, θ) of the first kind. If the g_φ-dual vector field $\theta^\sharp$ of θ belongs to $\mathfrak{aut}(M, \varphi)$, then*

(1) M is the total space of a fiber bundle over $\mathbb{S}^1$, and each fiber is endowed with a coupled SU(3)*-structure;*
(2) M has an LCC G_2-structure $\hat{\varphi}$ such that $d\hat{\varphi} = \hat{\theta} \wedge \hat{\varphi}$, where $\hat{\theta}$ is a 1-form with integral periods.

Motivated by the structure results for locally conformal symplectic structures of the first kind obtained in [1,15], we state the following more general problem.

Question 1. *What can one say about the structure of a (compact) 7-manifold M endowed with an LCC G_2-structure of the first kind?*

We conclude this section by mentioning a mild issue related to the above statement. In order to prove Theorem 1, one first deforms the Lee form of the given LCC G_2-structure on M to a closed 1-form with integral periods. Then, by a result of Tischler [29], M is the mapping torus N_ν of a compact 6-manifold N and a diffeomorphism $\nu\colon N \to N$, and one shows that N is endowed with a coupled SU(3)-structure (ω, ψ). However, in general, (ω, ψ) is not preserved by ν. In particular, it is not clear whether N_ν admits LCC G_2-structures arising from the mapping torus construction. A similar issue appears in locally conformal symplectic geometry. In [1], Banyaga proved that a compact manifold M endowed with an LCS structure of the first kind (ω, θ) is the total space of a mapping torus fiber bundle $P \to M = P_\varrho \to \mathbb{S}^1$ of a compact contact manifold (P, η) and a diffeomorphism $\varrho\colon P \to P$, which need not preserve the contact form η (if it does, then one can show that P_ϱ admits a natural LCS structure of the first kind). A different approach, which does not deform the given structure, was taken in [15]: the authors showed that, if (M, ω, θ) is a compact LCS manifold of the first kind and the codimension-one foliation given by the kernel of θ has a compact leaf, then M is diffeomorphic to the mapping torus P_ϱ of a compact contact manifold (P, η) and a strict contactomorphism $\varrho\colon P \to P$, and, moreover, the LCS structure (ω, θ) on M is precisely the one given by the mapping torus construction.

5. Lie Algebras with an LCC G_2-Structure

We begin this section recalling a few basic facts on Lie algebras, in order to introduce some notations. Then, we focus on the construction of Lie algebras admitting an LCC G_2-structure, and we prove a structure result for Lie algebras with an exact LCC G_2-structure. All Lie algebras considered in this section are assumed to be real.

5.1. Rank-One Extension of Lie Algebras

Let $\mathfrak{h}$ be a Lie algebra of dimension n, and denote by $[\cdot,\cdot]_\mathfrak{h}$ its Lie bracket and by $d_\mathfrak{h}$ the corresponding Chevalley–Eilenberg differential. The structure equations of $\mathfrak{h}$ with respect to a basis $(e_1, \ldots, e_n)$ are given by

$$[e_i, e_j]_\mathfrak{h} = \sum_{k=1}^{n} c_{ij}^k e_k, \qquad 1 \le i < j \le n,$$

with $c_{ij}^k \in \mathbb{R}$, $c_{ij}^k = -c_{ji}^k$, and $\sum_{r=1}^{n} \left(c_{ij}^r c_{rk}^s + c_{jk}^r c_{ri}^s + c_{ki}^r c_{rj}^s \right) = 0$. Equivalently, if $(e^1, \ldots, e^n)$ is the dual basis of $(e_1, \ldots, e_n)$, then the structure equations of $\mathfrak{h}$ can be written as follows:

$$d_{\mathfrak{h}} e^k = - \sum_{1 \le i < j \le n} c_{ij}^k e^i \wedge e^j, \qquad 1 \le k \le n.$$

A Lie algebra $\mathfrak{h}$ is then described up to isomorphism by the n-tuple $(d_{\mathfrak{h}} e^1, \ldots, d_{\mathfrak{h}} e^n)$.

The *rank-one extension* of $\mathfrak{h}$ induced by a derivation $D \in \mathrm{Der}(\mathfrak{h})$ is the $(n+1)$-dimensional Lie algebra given by the vector space $\mathfrak{h} \oplus \mathbb{R}$ endowed with the Lie bracket

$$[(X, a), (Y, b)] := ([X, Y]_{\mathfrak{h}} + a\, D(Y) - b\, D(X), 0)$$

for all (X, a), $(Y, b) \in \mathfrak{h} \oplus \mathbb{R}$. We shall denote this Lie algebra by $\mathfrak{h} \rtimes_D \mathbb{R}$. Moreover, we let $\xi := (0, 1)$, and we denote by η the 1-form on $\mathfrak{h} \rtimes_D \mathbb{R}$ such that $\eta(\xi) = 1$ and $\eta(X) = 0$, for all $X \in \mathfrak{h}$. Notice that, if $\mathfrak{h}$ is a nilpotent Lie algebra and D is a nilpotent derivation, then $\mathfrak{h} \rtimes_D \mathbb{R}$ is nilpotent.

Let d denote the Chevalley–Eilenberg differential on $\mathfrak{h} \rtimes_D \mathbb{R}$. Using the Koszul formula, it is possible to check that for every k-form $\gamma \in \Lambda^k(\mathfrak{h}^*)$, the following identity holds:

$$d\gamma = d_{\mathfrak{h}} \gamma + (-1)^{k+1} D^* \gamma \wedge \eta \tag{4}$$

where the natural action of an endomorphism $A \in \mathrm{End}(\mathfrak{h})$ on $\Lambda^k(\mathfrak{h}^*)$ is given by

$$A^* \gamma(X_1, \ldots, X_k) = \gamma(AX_1, \ldots, X_k) + \cdots + \gamma(X_1, \ldots, AX_k)$$

for all $X_1, \ldots, X_k \in \mathfrak{h}$. Moreover, it is clear that $d\eta = 0$.

5.2. A Structure Result for Lie Algebras with an Exact LCC G_2-Structure

Let $\mathfrak{h}$ be a six-dimensional Lie algebra. A pair $(\omega, \psi) \in \Lambda^2(\mathfrak{h}^*) \times \Lambda^3(\mathfrak{h}^*)$ defines an SU(3)-structure on $\mathfrak{h}$ if there exists a basis $(e^1, \ldots, e^6)$ of $\mathfrak{h}^*$ such that

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi = e^{135} - e^{146} - e^{236} - e^{245}. \tag{5}$$

We shall call $(e^1, \ldots, e^6)$ an SU(3)-*basis* for $(\mathfrak{h}, \omega, \psi)$. An SU(3)-structure (ω, ψ) on $\mathfrak{h}$ is *half-flat* if $d_{\mathfrak{h}} \omega \wedge \omega = 0$ and $d_{\mathfrak{h}} \psi = 0$. A half-flat SU(3)-structure satisfying the condition $d_{\mathfrak{h}} \omega = c\psi$ for some $c \in \mathbb{R}$ is *coupled* if $c \neq 0$, while it is *symplectic half-flat* if $c = 0$.

Similarly, a 3-form φ on a seven-dimensional Lie algebra $\mathfrak{g}$ defines a G_2-structure if there is a basis $(e^1, \ldots, e^7)$ of $\mathfrak{g}^*$ such that

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

We shall refer to $(e^1, \ldots, e^7)$ as a G_2-*basis* for $(\mathfrak{g}, \varphi)$. A G_2-structure φ is *closed* if $d_{\mathfrak{g}} \varphi = 0$, while it is *locally conformal closed* (*LCC*) if $d_{\mathfrak{g}} \varphi = \theta \wedge \varphi$ for some 1-form $\theta \in \Lambda^1(\mathfrak{g}^*)$ with $d_{\mathfrak{g}} \theta = 0$.

If $\mathfrak{h} \rtimes_D \mathbb{R}$ is the rank-one extension of a six-dimensional Lie algebra $\mathfrak{h}$ endowed with an SU(3)-structure (ω, ψ), then it admits a G_2-structure defined by the 3-form

$$\varphi = \omega \wedge \eta + \psi.$$

Indeed, if $(e^1, \ldots, e^6)$ is an SU(3)-basis for $(\mathfrak{h}, \omega, \psi)$, then $(e^1, \ldots, e^6, e^7)$ with $e^7 := \eta$ is a G_2-basis for $(\mathfrak{h} \rtimes_D \mathbb{R}, \varphi)$.

In the next proposition, we collect some conditions guaranteeing the existence of an LCC G_2-structure on the rank-one extension of a six-dimensional Lie algebra. For the sake of convenience, from now on, we shall denote the Chevalley–Eilenberg differential on seven-dimensional Lie algebras simply by d.

Proposition 3. *Let $\mathfrak{h}$ be a six-dimensional Lie algebra endowed with a coupled* $\mathrm{SU}(3)$*-structure (ω, ψ) with $d_{\mathfrak{h}}\omega = c\psi$, and consider the rank-one extension $\mathfrak{h} \rtimes_D \mathbb{R}$, $D \in \mathrm{Der}(\mathfrak{h})$, endowed with the G_2-structure $\varphi := \omega \wedge \eta + \psi$. Then, the following holds:*

(i) *φ is LCC with Lee form $\theta = a\eta$, for some $a \in \mathbb{R}$, if and only if $D^*\psi = -(a+c)\psi$. In particular, it is closed if and only if $D^*\psi = -c\,\psi$.*

(ii) *If $D^*\omega = \mu\omega$ with $\mu \neq -c$, then φ is $d_{(-(c+\mu)\eta)}$-exact. Moreover, it is of the first kind if and only if $\mu = 0$.*

Proof. Using Equation (4), we see that the G_2-structure $\varphi = \omega \wedge \eta + \psi$ is LCC with Lee form $\theta = a\eta$ if and only if

$$a\eta \wedge \psi = a\eta \wedge \varphi = d(\omega \wedge \eta + \psi) = d_{\mathfrak{h}}\omega \wedge \eta + d_{\mathfrak{h}}\psi + D^*\psi \wedge \eta = (c\psi + D^*\psi) \wedge \eta.$$

From this, (i) follows.

As for (ii), we first observe that the hypothesis $D^*\omega = \mu\omega$ implies

$$D^*\psi = \frac{1}{c}\, D^* d_{\mathfrak{h}}\omega = \frac{1}{c}\, d_{\mathfrak{h}} D^*\omega = \frac{\mu}{c}\, d_{\mathfrak{h}}\omega = \mu\psi.$$

Thus, φ is LCC with Lee form $\theta = -(c+\mu)\eta$ by Point *(i)*. Moreover,

$$d\omega = d_{\mathfrak{h}}\omega - D^*\omega \wedge \eta = c\psi - \mu\omega \wedge \eta.$$

Consequently,

$$\varphi = \omega \wedge \eta + \psi = \omega \wedge \eta + \frac{1}{c}\,(d\omega + \mu\omega \wedge \eta) = d\left(\frac{\omega}{c}\right) + (c+\mu)\eta \wedge \frac{\omega}{c}.$$

Hence, $\varphi = d_{(-(c+\mu)\eta)}\left(\frac{\omega}{c}\right)$ is exact. Notice that $\frac{\omega}{c} = \iota_{\frac{\xi}{c}}\varphi \in \Lambda^2_7((\mathfrak{h} \rtimes_D \mathbb{R})^*)$. Therefore, according to Proposition 2, φ is of the first kind if and only if

$$0 = \theta\left(\frac{\xi}{c}\right) + 1 = -(c+\mu)\,\eta\left(\frac{\xi}{c}\right) + 1 = -\frac{\mu}{c}.$$

□

Remark 3.

(1) *Proposition 3 generalizes some results obtained by the second author in the joint works [12,14]. In detail, [12] (Proposition 5.1) corresponds to Point (i) with $a = -c$, while [14] (Proposition 4.2) corresponds to Point (i) with $a = c$.*

(2) *When the SU(3)-structure (ω, ψ) on $\mathfrak{h}$ is symplectic half-flat and $D \in \mathrm{Der}(\mathfrak{h})$ satisfies $D^*\psi = 0$, then $\varphi = \omega \wedge \eta + \psi$ is a closed G_2-structure on $\mathfrak{h} \rtimes_D \mathbb{R}$ by Point (i). This was already observed by Manero in [30] (Proposition 1.1).*

(3) *Recall that for a six-dimensional Lie algebra $\mathfrak{h}$ endowed with an SU(3)-structure (ω, ψ), the following isomorphisms hold:*

$$\{A \in \mathrm{End}(\mathfrak{h}) \mid A^*\omega = 0\} \cong \mathfrak{sp}(6, \mathbb{R}), \quad \{A \in \mathrm{End}(\mathfrak{h}) \mid A^*\psi = 0\} \cong \mathfrak{sl}(3, \mathbb{C}) \subset \mathfrak{gl}(6, \mathbb{R}).$$

In particular, if (ω, ψ) is coupled and $A^*\omega = 0$, then $A \in \mathfrak{sp}(6, \mathbb{R}) \cap \mathfrak{sl}(3, \mathbb{C}) = \mathfrak{su}(3)$.

The next result is the converse of Point (ii) of Proposition 3.

Proposition 4. *Let $\mathfrak{g}$ be a seven-dimensional Lie algebra endowed with an exact LCC G_2-structure $\varphi = d\sigma - \theta \wedge \sigma$, where $\theta \in \Lambda^1(\mathfrak{g}^*)$ is closed and $\sigma \in \Lambda^2_7(\mathfrak{g}^*)$. Assume that the non-zero vector $X \in \mathfrak{g}$ for which $\sigma = \iota_X\varphi$ satisfies $\theta(X) \neq 0$. Then, $\mathfrak{g}$ splits as a g_φ-orthogonal direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$, where $\mathbb{R} = \langle X \rangle$ and $\mathfrak{h} := \ker(\theta)$ is a six-dimensional ideal endowed with a coupled $\mathrm{SU}(3)$-structure (ω, ψ) induced by φ. Moreover, there is a derivation $D \in \mathrm{Der}(\mathfrak{h})$ such that $D^*\omega = -(1+\theta(X))\,\omega$, and $\mathfrak{g} \cong \mathfrak{h} \rtimes_D \mathbb{R}$.*

Proof. It is clear that $\mathfrak{h} := \ker(\theta)$ is a six-dimensional ideal of $\mathfrak{g}$, as $\theta \in \Lambda^1(\mathfrak{g}^*)$ is non-zero and closed. Since $\theta(X) \neq 0$, we see that the vector space $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$, with $\mathbb{R} = \langle X \rangle$. The $\mathbb{R}$-linear map

$$D : \mathfrak{h} \to \mathfrak{h}, \quad H \mapsto [X, H]$$

is well-defined, as $d\theta = 0$, and it is a derivation of $\mathfrak{h}$ by the Jacobi identity. From this, it is easy to see that $\mathfrak{g} \cong \mathfrak{h} \rtimes_D \mathbb{R}$ as a Lie algebra.

Let $\theta^\sharp \in \mathfrak{g}$ be the g_φ-dual vector of θ. By definition, $\theta(\theta^\sharp) = g_\varphi(\theta^\sharp, \theta^\sharp) = |\theta|^2 \neq 0$. Thus, $\theta^\sharp \in \langle X \rangle \subset \mathfrak{g}$ and the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ is g_φ-orthogonal, i.e., $g_\varphi(H, X) = 0$ for all $H \in \mathfrak{h}$. Consequently, depending on the choice of a unit vector $\varepsilon \frac{X}{|X|} \in \langle X \rangle$, with $\varepsilon \in \{\pm 1\}$, the ideal $\mathfrak{h}$ admits an SU(3)-structure defined by the pair

$$\omega := \left(\iota_{\varepsilon \frac{X}{|X|}} \varphi \right) \Big|_{\mathfrak{h}}, \quad \psi := \varphi|_{\mathfrak{h}}.$$

Notice that $\omega = \varepsilon |X|^{-1} \sigma\big|_{\mathfrak{h}} = \varepsilon |X|^{-1}\sigma$, as $\iota_X \sigma = 0$. We claim that (ω, ψ) is coupled with coupling constant $c = \varepsilon |X|^{-1}$. First, observe that for all $H_1, H_2, H_3 \in \mathfrak{h}$, we have

$$\psi(H_1, H_2, H_3) = (d\sigma - \theta \wedge \sigma)(H_1, H_2, H_3) = d\sigma(H_1, H_2, H_3) = d_{\mathfrak{h}}\sigma(H_1, H_2, H_3).$$

Therefore, $d_{\mathfrak{h}}\omega = \varepsilon |X|^{-1}\psi$, and the claim is proved. Let us now determine the expression of $(D^*\sigma)|_{\mathfrak{h}}$, from which we will deduce the expression of $D^*\omega$. For all $H_1, H_2 \in \mathfrak{h}$, we have

$$D^*\sigma(H_1, H_2) = \sigma([X, H_1], H_2) - \sigma([X, H_2], H_1) = -d\sigma(X, H_1, H_2) = -(\iota_X d\sigma)(H_1, H_2)$$

where the second equality follows from Koszul formula and the condition $\iota_X \sigma = 0$. Since $\varphi = d\sigma - \theta \wedge \sigma$, on $\mathfrak{h}$ we have

$$D^*\sigma = -\iota_X d\sigma = -\iota_X(\varphi + \theta \wedge \sigma) = -(1 + \theta(X))\,\sigma.$$

Thus,

$$D^*\omega = \varepsilon |X|^{-1} D^*\sigma = -(1 + \theta(X))\,\omega.$$

□

Combining Propositions 3 and 4, we obtain the following analogue of [28] (Theorem 1.4) for exact locally conformal symplectic Lie algebras.

Theorem 2. *There is a one-to-one correspondence between seven-dimensional Lie algebras $\mathfrak{g}$ admitting an exact G_2-structure of the form $\varphi = d\sigma - \theta \wedge \sigma$, with $\sigma = \iota_X\varphi \in \Lambda^2_7(\mathfrak{g}^*)$ and $\theta(X) \neq 0$, and six-dimensional Lie algebras $\mathfrak{h}$ endowed with a coupled $\mathrm{SU}(3)$-structure (ω, ψ), with coupling constant c, and a derivation $D \in \mathrm{Der}(\mathfrak{h})$ such that $D^*\omega = \mu\omega$, for some $\mu \neq -c$.*

Remark 4. *Comparing Theorem 2 with Theorem 1, we see that in the former we do not have any issue with deformations. Indeed, the ideal of $\mathfrak{g}$ admitting a coupled SU(3)-structure is precisely the kernel of the Lee form θ, while the fibration considered in Theorem 1 is associated with a closed 1-form arising from a deformation of the Lee form.*

According to a result of Dixmier (see [18] (Theorem 1)), the Lichnerowicz cohomology of a nilpotent Lie algebra with respect to any closed 1-form vanishes. Hence, every LCC G_2-structure on a seven-dimensional nilpotent Lie algebra is exact. We use this observation to prove the following result.

Proposition 5. *None of the seven-dimensional non-Abelian nilpotent Lie algebras admitting closed G_2-structures admits LCC G_2-structures.*

Proof. By the classification result of Conti-Fernández [17], a seven-dimensional non-Abelian nilpotent Lie algebra admitting closed G_2-structures is isomorphic to one of the following:

$$
\begin{aligned}
\mathfrak{n}_1 &= (0,0,0,0,e^{12},e^{13},0),\\
\mathfrak{n}_2 &= (0,0,0,e^{12},e^{13},e^{23},0),\\
\mathfrak{n}_3 &= (0,0,e^{12},0,0,e^{13}+e^{24},e^{15}),\\
\mathfrak{n}_4 &= (0,0,e^{12},0,0,e^{13},e^{14}+e^{25}),\\
\mathfrak{n}_5 &= (0,0,0,e^{12},e^{13},e^{14},e^{15}),\\
\mathfrak{n}_6 &= (0,0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}),\\
\mathfrak{n}_7 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}),\\
\mathfrak{n}_8 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}+e^{25}),\\
\mathfrak{n}_9 &= (0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23}),\\
\mathfrak{n}_{10} &= (0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{15}+e^{16}-3e^{26}),\\
\mathfrak{n}_{11} &= (0,0,0,e^{12},e^{23},-e^{13},2e^{26}-2e^{34}-2e^{16}+2e^{25}).
\end{aligned}
$$

To show the proposition, we will use Dixmier's result together with the following fact: a 3-form ϕ on a seven-dimensional Lie algebra $\mathfrak{g}$ defines a G_2-structure if and only if the symmetric bilinear map

$$
b_\phi : \mathfrak{g}\times\mathfrak{g} \to \Lambda^7(\mathfrak{g}^*) \cong \mathfrak{g}, \quad (X,Y) \mapsto \frac{1}{6}\,\iota_X\phi\wedge\iota_Y\phi\wedge\phi
$$

is definite (cf. [21]). Now, for every nilpotent Lie algebra $\mathfrak{n}_i$ appearing above, we consider the generic closed 1-form $\theta = \sum_{k=1}^{7}\theta_k e^k \in \Lambda^1(\mathfrak{n}_i^*)$, with some of the real numbers θ_k possibly zero as $d\theta = 0$, and the generic d_θ-exact 3-form $\phi = d\sigma - \theta\wedge\sigma$, where $\sigma = \sum_{1\le j<k\le 7}\sigma_{jk}e^{jk} \in \Lambda^2(\mathfrak{n}_i^*)$. Then, we compute the map b_ϕ associated with such a 3-form ϕ, and we observe that in each case it cannot be definite. Indeed, it is just a matter of computation to show that $b_\phi(e_6,e_6) = 0$ for the nilpotent Lie algebras $\mathfrak{n}_i$, with $i = 1,2,3,4,5,6$ and that $b_\phi(e_7,e_7) = 0$ for the remaining ones. □

6. Examples

We now use the results of the previous section to construct various examples of LCC G_2-structures that clarify the interplay between the conditions discussed in Sections 3 and 4.

First of all, we need to start with a six-dimensional Lie algebra admitting coupled SU(3)-structures. In the nilpotent case, the following classification is known (see [14] (Theorem 4.1)).

Theorem 3 ([14]). *Up to isomorphism, a six-dimensional non-Abelian nilpotent Lie algebra admitting coupled* SU(3)*-structures is isomorphic to one of the following*

$$
\mathfrak{h}_1 = \left(0,0,0,0,e^{14}+e^{23},e^{13}-e^{24}\right), \quad \mathfrak{h}_2 = \left(0,0,0,e^{13},e^{14}+e^{23},e^{13}-e^{15}-e^{24}\right).
$$

In both cases, $(e^1, \dots, e^6)$ is an SU(3)*-basis for a certain coupled structure (ω, ψ).*

Let us consider the coupled SU(3)-structure (ω, ψ) on $\mathfrak{h}_1$. Since $(e^1, \dots, e^6)$ is an SU(3)-basis, the forms ω and ψ can be written as in Equation (5), and a simple computation shows that $d_{\mathfrak{h}_1}\omega = -\psi$. As observed in [14], the inner product $g = \sum_{i=1}^{6}(e^i)^2$ induced by (ω, ψ) is a *nilsoliton*, i.e., its Ricci operator is of the form

$$\mathrm{Ric}(g) = -3\,\mathrm{Id} + 4D_1 \tag{6}$$

where $D_1 \in \mathrm{Der}(\mathfrak{h}_1)$ is given by

$$D_1(e_1) = \frac{1}{2}e_1,\ D_1(e_2) = \frac{1}{2}e_2,\ D_1(e_3) = \frac{1}{2}e_3,\ D_1(e_4) = \frac{1}{2}e_4,\ D_1(e_5) = e_5,\ D_1(e_6) = e_6,$$

$(e_1, \dots, e_6)$ being the basis of $\mathfrak{h}_1$ whose dual basis is the SU(3)-basis of $(\mathfrak{h}_1, \omega, \psi)$. For more details on nilsolitons we refer the reader to [31].

We know that the rank-one extension $\mathfrak{h}_1 \rtimes_D \mathbb{R}$ of $\mathfrak{h}_1$ induced by a derivation $D \in \mathrm{Der}(\mathfrak{h}_1)$ admits a G_2-structure defined by the 3-form $\varphi = \omega \wedge \eta + \psi$ and that the G_2-basis is given by $(e^1, \dots, e^6, e^7)$ with $e^7 := \eta$. In what follows, we shall always write the structure equations of $\mathfrak{h}_1 \rtimes_D \mathbb{R}$ with respect to such a basis.

The first example we consider was discussed in [14]. It consists of a solvable Lie algebra endowed with an LCC G_2-structure φ inducing an Einstein inner product. As we will see, φ is not exact, that is, its class $[\varphi]_\theta$ in the Lichnerowicz cohomology is not zero.

Example 1. *Let us consider the derivation $D_1 \in \mathrm{Der}(\mathfrak{h}_1)$ appearing in Equation (6). The rank-one extension $\mathfrak{h}_1 \rtimes_{D_1} \mathbb{R}$ of $\mathfrak{h}_1$ has structure equations*

$$\left(\frac{1}{2}e^{17}, \frac{1}{2}e^{27}, \frac{1}{2}e^{37}, \frac{1}{2}e^{47}, e^{14} + e^{23} + e^{57}, e^{13} - e^{24} + e^{67}, 0\right).$$

Since $D_1^*\psi = 2\psi$ and the coupling constant is $c = -1$, the G_2-structure $\varphi = \omega \wedge \eta + \psi$ on $\mathfrak{h}_1 \rtimes_{D_1} \mathbb{R}$ is LCC with Lee form $\theta = -\eta$, by Point (i) of Proposition 3. Moreover, it induces the inner product $g_\varphi = g + \eta^2$, which is Einstein with Ricci operator $\mathrm{Ric}(g_\varphi) = -3\,\mathrm{Id}$ by [32] (Lemma 2). A simple computation shows that φ cannot be equal to $d_\theta\sigma$ for any 2-form $\sigma \in \Lambda^2((\mathfrak{h}_1 \rtimes_{D_1} \mathbb{R})^*)$. In particular, it is of the second kind.

We conclude this example observing that the Lie algebra $\mathfrak{h}_1 \rtimes_{D_1} \mathbb{R}$ is solvable and not unimodular, as $\mathrm{tr}(\mathrm{ad}_{e_7}) = \mathrm{tr}(D_1) = 4$. Thus, the corresponding simply connected solvable Lie group does not admit any compact quotient.

The next two examples were obtained in [12] (Section 5). In the first one, the LCC G_2-structure is of the first kind, while in the second one the LCC G_2-structure is exact but it is not of the first kind.

Example 2. *Consider the derivation $D_2 \in \mathrm{Der}(\mathfrak{h}_1)$ defined as follows:*

$$D_2(e_1) = -e_3,\ D_2(e_2) = -e_4,\ D_2(e_3) = e_1,\ D_2(e_4) = e_2,\ D_2(e_5) = 0,\ D_2(e_6) = 0.$$

Then, the rank-one extension $\mathfrak{h}_1 \rtimes_{D_2} \mathbb{R}$ has structure equations

$$\left(e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\right),$$

and $D_2^*\omega = 0$. Thus, by Point (ii) of Proposition 3, we have that the 3-form $\varphi = \omega \wedge \eta + \psi$ defines an LCC G_2-structure of the first kind on $\mathfrak{h}_1 \rtimes_{D_2} \mathbb{R}$ with Lee form $\theta = \eta$.

Example 3. *Consider the rank-one extension $\mathfrak{h}_1 \rtimes_{D_3} \mathbb{R}$, where $D_3 \in \mathrm{Der}(\mathfrak{h}_1)$ is given by*

$$D_3(e_1) = 2e_3,\ D_3(e_2) = 2e_4,\ D_3(e_3) = e_1,\ D_3(e_4) = e_2,\ D_3(e_5) = 0,\ D_3(e_6) = 0.$$

The structure equations of $\mathfrak{h}_1 \rtimes_{D_3} \mathbb{R}$ are the following:

$$\left(e^{37}, e^{47}, 2e^{17}, 2e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\right).$$

Since $D_3^*\psi = 0$ but $D_3^*\omega \neq 0$, the G_2-structure $\varphi = \omega \wedge \eta + \psi$ on $\mathfrak{h}_1 \rtimes_{D_3} \mathbb{R}$ is LCC with Lee form $\theta = \eta$, by Point (i) of Proposition 3. We observe that

$$\varphi = d_\theta \gamma$$

where $\gamma = \frac{5}{7}e^{12} - \frac{3}{7}e^{14} + \frac{3}{7}e^{23} - \frac{1}{7}e^{34} - e^{56}$ does not belong to $\Lambda^2_7((\mathfrak{h}_1 \rtimes_{D_3} \mathbb{R})^*)$. In this case, the only infinitesimal automorphisms of φ are of the form $X = a\, e_5 + b\, e_6 \in \mathfrak{h}_1 \rtimes_{D_3} \mathbb{R}$, with $a, b \in \mathbb{R}$. Thus, φ is of the second kind.

Remark 5. *As shown in [12], the Lie algebras considered in Examples 2 and 3 are solvable and unimodular, and the corresponding simply connected solvable Lie groups admit a lattice. Thus, both examples give rise to a compact seven-dimensional solvmanifold endowed with an LCC G_2-structure.*

Remark 6. *It was proved in [15] (Proposition 5.5) that, on a unimodular Lie algebra, every exact locally conformal symplectic structure is of the first kind. This is not the case in the G_2 setting: indeed, the LCC G_2-structure of Example 3 is exact but not on the first kind, while the Lie algebra $\mathfrak{h}_1 \rtimes_{D_3} \mathbb{R}$ is unimodular.*

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